807 HW5

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1. u satisfies

$$u_{xx} + u_{yy} = 0 \quad \text{in} \quad \Omega = (-1,1) \times (-1,1)$$

, has known Dirichlet boundary condition, and is symmetrical with respect to the axis x and y. Write down the simplest finite-difference equations for $dx = dy = \frac{1}{2}$. Prove that the Jacobi and Gauss-Seidal iterative processes for their solution both converge. Hence verify that the asymptotic rate of convergence of the Gauss-Seidal iteration is twice that of the Jacobi iteration.

2. u satisfies the Poisson's equation at the points of rectangle $0 < x < ph$, $0 < y < qh$ and has known values on its boundary. Write down the matrix of the five-point difference equations approximating this problem at the mesh points defined by $x = ih$, $y = jh$, $i = 1 : 1 : p-1$, $j = 1 : 1 : q-1$. Show that the eigenvalue $\lambda_{i,j}$ of the matrix are given by

$$\lambda_{i,j} = -4 + 2\left(\cos\left(\frac{i\pi}{p}\right) + \cos\left(\frac{j\pi}{q}\right)\right)$$

Deduce that the spectral radius of the corresponding Jacobi iteration matrix is

$$\frac{1}{2}\left(\cos\left(\frac{\pi}{p}\right) + \cos\left(\frac{\pi}{q}\right)\right)$$

3. u satisfies

$$u_t = u_{xx}, \quad 0 < x < 1,$$

the initial condition $u = 1$ when $t = 0$, $0 < x < 1$, and the boundary condition $u = 0$ at $x = 0$ and 1, $t \geq 0$. Approximate the difference equation by Crank-Nicolson equations taking $dx = 0.1$ and $r = \frac{dt}{dx^2} = 1$. Write a code to solve them for one time-step by the Jacobi, Gauss-Seidal, and SOR iterative methods, taking $\omega = 1.064$ and the first approximation values equal to the initial values. Show the results for the first ten iterations for the one time-step. What do you observe from the results?

4. Write a code to solve

$$u_{xx} + u_{yy} + 2 = 0$$

over the rectangular domain $0 < x < 2$, $0 < y < 1$ where $u = 0$ on the boundary by Gauss-Jacobi Method on five point discretization until four digits after decimal points are the same. ($dx = dy = \frac{1}{4}$). From problem 2, we know the spectral radius of the Jacobi iteration matrix. Use Lyusternik's method to estimate λ_1 and the solution by the values obtained from 9th, 10th and 11th iteration of Gauss-Jacobi method. Use Aitken's method to estimate the solution by the values obtained from 9th, 10th and 11th iteration of Gauss-Jacobi method. Use both of these methods to give an estimation of the solution. (refer to Table 5.4 in the book)

5.

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & 0 & a_{25} \\ 0 & a_{32} & a_{33} & 0 & 0 \\ a_{41} & 0 & 0 & a_{44} & a_{45} \\ 0 & a_{52} & 0 & a_{54} & a_{55} \end{bmatrix}$$

(a) show that the matrix A is 2-cyclic.

(b) write down two ordering vectors from A in terms of the numbers 0 and 1. Is A consistently ordered with respect to either of them? If not, use one of them to re-order A into a consistently ordered matrix B.

(c) Verify that A is consistenctly ordered w.r.t to the ordering vector $\gamma^{(3)} = (0, 1, 2, 1, 2)$. Use $\gamma^{(3)}$ to re-ordered A into a consistently ordered block tridiagonal matrix C. Write down, in matrix notation, the equation giving the re-ordering of Ax=b corresponding to C.

6. Write the usual notation show that the eigenvalues of the Jacobi and SOR iteration matrices are respectively the roots of the equations

(i) $\det(\mu D - L - U) = 0$

(ii) $\det(kD - \lambda\omega L - \omega U) = 0$, where $k = \lambda + \omega - 1$.

7. From the results of Problem 6, show that the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ -1 & 0 & 0 & 4 \end{bmatrix}$$

is consistntly ordered w.r.t the ordering vector $(0, 1, 2, 1)$. Prove, for this particular matrix, that the eigenvalues λ of the associated SOR iteration matrix are related to the eigenvalues μ of the corresponding Jacobi iteration matrix by $(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2$. Comment on this result.