Boyce/DiPrima 9th ed, Ch 7.3: Systems of Linear Equations, Linear Independence, Eigenvalues

- A system of n linear equations in n variables,

$$\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n ,
\end{aligned}$$

can be expressed as a matrix equation $\mathbf{Ax} = \mathbf{b}$:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

- If $\mathbf{b} = \mathbf{0}$, then system is **homogeneous**; otherwise it is **nonhomogeneous**.

Nonsingular Case

- If the coefficient matrix $\mathbf{A}$ is nonsingular, then it is invertible and we can solve $\mathbf{Ax} = \mathbf{b}$ as follows:

$$\mathbf{Ax} = \mathbf{b} \;\Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

- This solution is therefore unique. Also, if $\mathbf{b} = \mathbf{0}$, it follows that the unique solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}$.

- Thus if $\mathbf{A}$ is nonsingular, then the only solution to $\mathbf{Ax} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$.

Example 1: Nonsingular Case (1 of 3)

- From a previous example, we know that the matrix **A** below is nonsingular with inverse as given.

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} -3/4 & -5/4 & 1/4 \\ -5/4 & -7/4 & -1/4 \\ -1/4 & -3/4 & -1/4 \end{pmatrix}$$

- Using the definition of matrix multiplication, it follows that the only solution of $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \begin{pmatrix} -3/4 & -5/4 & 1/4 \\ -5/4 & -7/4 & -1/4 \\ -1/4 & -3/4 & -1/4 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 1: Nonsingular Case (2 of 3)

- Now let's solve the nonhomogeneous linear system **Ax** = **b** below using $\mathbf{A}^{-1}$:

$$\begin{aligned} 0x_1 + x_2 + 2x_3 &= 2 \\ 1x_1 + 0x_2 + 3x_3 &= -2 \\ 4x_1 - 3x_2 + 8x_3 &= 0 \end{aligned}$$

- This system of equations can be written as **Ax** = **b**, where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix}$$

- Then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} -3/4 & -5/4 & 1/4 \\ -5/4 & -7/4 & -1/4 \\ -1/4 & -3/4 & -1/4 \end{pmatrix} \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Example 1: Nonsingular Case (3 of 3)

- Alternatively, we could solve the nonhomogeneous linear system **Ax** = **b** below using row reduction.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ -x_1 + x_2 - 2x_3 &= -5 \\ 2x_1 - x_2 - x_3 &= 4 \end{aligned}$$

- To do so, form the augmented matrix (**A**|**b**) and reduce, using elementary row operations.

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & -7 & -10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{aligned} x_1 - 2x_2 + 3x_3 &= 7 \\ x_2 - x_3 &= -2 \\ x_3 &= 1 \end{aligned} \to \mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Singular Case

- If the coefficient matrix $\mathbf{A}$ is singular, then $\mathbf{A}^{-1}$ does not exist, and either a solution to $\mathbf{Ax} = \mathbf{b}$ does not exist, or there is more than one solution (not unique).
- Further, the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has more than one solution. That is, in addition to the trivial solution $\mathbf{x} = \mathbf{0}$, there are infinitely many nontrivial solutions.
- The nonhomogeneous case $\mathbf{Ax} = \mathbf{b}$ has no solution unless $(\mathbf{b}, \mathbf{y}) = 0$, for all vectors $\mathbf{y}$ satisfying $\mathbf{A}^*\mathbf{y} = \mathbf{0}$, where $\mathbf{A}^*$ is the adjoint of $\mathbf{A}$.
- In this case, $\mathbf{Ax} = \mathbf{b}$ has solutions (infinitely many), each of the form $\mathbf{x} = \mathbf{x}^{(0)} + \boldsymbol{\xi}$, where $\mathbf{x}^{(0)}$ is a particular solution of $\mathbf{Ax} = \mathbf{b}$, and $\boldsymbol{\xi}$ is any solution of $\mathbf{Ax} = \mathbf{0}$.

Example 2: Singular Case (1 of 2)

- Solve the nonhomogeneous linear system **Ax** = **b** below using row reduction. Observe that the coefficients are nearly the same as in the previous example

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= b_1 \\ -x_1 + x_2 - 2x_3 &= b_2 \\ 2x_1 - x_2 + 3x_3 &= b_3 \end{aligned}$$

- We will form the augmented matrix (**A**|**b**) and use some of the steps in Example 1 to transform the matrix more quickly

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & -2 & 3 & b_1 \\ -1 & 1 & -2 & b_2 \\ 2 & -1 & 3 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{pmatrix}$$

$$\to \begin{aligned} x_1 - 2x_2 + 3x_3 &= b_1 \\ x_2 - x_3 &= -b_1 - b_2 \\ 0 &= b_1 + 3b_2 + b_3 \end{aligned} \quad \to b_1 + 3b_2 + b_3 = 0$$

$$x_1 - 2x_2 + 3x_3 = b_1$$
$$-x_1 + x_2 - 2x_3 = b_2$$
$$2x_1 - x_2 + 3x_3 = b_3$$

Example 2: Singular Case (2 of 2)

- From the previous slide, if $b_1 + 3b_2 + b_3 \neq 0$, there is no solution to the system of equations
- Requiring that $b_1 + 3b_2 + b_3 = 0$, assume, for example, that
$$b_1 = 2,\ b_2 = 1,\ b_3 = -5$$
- Then the reduced augmented matrix (**A**|**b**) becomes:

$$\begin{pmatrix} 1 & -2 & 3 & b_1 \\ 0 & 1 & -1 & -b_1 - b_2 \\ 0 & 0 & 0 & b_1 + 3b_2 + b_3 \end{pmatrix} \to \begin{array}{rrrl} x_1 & -2x_2 & +3\,x_3 & = 2 \\ & x_2 & - \;\; x_3 & = -3 \\ & & 0 & = 0 \end{array} \to \mathbf{x} = \begin{pmatrix} -x_3 - 4 \\ x_3 - 3 \\ x_3 \end{pmatrix} \to \mathbf{x} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix}$$

- It can be shown that the second term in **x** is a solution of the nonhomogeneous equation and that the first term is the most general solution of the homogeneous equation, letting $x_3 = \alpha$, where α is arbitrary

Linear Dependence and Independence

- A set of vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)},\ldots, \mathbf{x}^{(n)}$ is **linearly dependent** if there exists scalars $c_1, c_2,\ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{0}$$

- If the only solution of

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{0}$$

  is $c_1 = c_2 = \ldots = c_n = 0$, then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)},\ldots, \mathbf{x}^{(n)}$ is **linearly independent.**

Example 3: Linear Dependence (1 of 2)

- Determine whether the following vectors are linear dependent or linearly independent.

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \ \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \ \mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$$

- We need to solve

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

or

$$c_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c\begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \ \mathbf{x}^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \ \mathbf{x}^{(3)} = \begin{pmatrix} -4 \\ 1 \\ -11 \end{pmatrix}$$

Example 3: Linear Dependence (2 of 2)

- We can reduce the augmented matrix $(\mathbf{A}|\mathbf{b})$, as before.

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & -11 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -4 & 0 \\ 0 & -3 & 9 & 0 \\ 0 & 5 & 15 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{array}{rrrl} c_1 & +2c_2 & -4c_3 & = 0 \\ & c_2 & -3c_3 & = 0 \\ & & 0 & = 0 \end{array} \to \mathbf{c} = c_3 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \text{ where } c_3 \text{ can be any number}$$

- So, the vectors are linearly dependent: if $c_3 = -1$, $2\mathbf{x}^{(1)} - 3\mathbf{x}^{(2)} - \mathbf{x}^{(3)} = \mathbf{0}$
- Alternatively, we could show that the following determinant is zero:

$$\det(x_{ij}) = \begin{vmatrix} 1 & 2 & -4 \\ 2 & 1 & 1 \\ -1 & 3 & -11 \end{vmatrix} = 0$$

Linear Independence and Invertibility

- Consider the previous two examples:
  - The first matrix was known to be nonsingular, and its column vectors were linearly independent.
  - The second matrix was known to be singular, and its column vectors were linearly dependent.
- This is true in general: the columns (or rows) of $\mathbf{A}$ are linearly independent iff $\mathbf{A}$ is nonsingular iff $\mathbf{A}^{-1}$ exists.
- Also, $\mathbf{A}$ is nonsingular iff $\det\mathbf{A} \neq 0$, hence columns (or rows) of $\mathbf{A}$ are linearly independent iff $\det\mathbf{A} \neq 0$.
- Further, if $\mathbf{C} = \mathbf{AB}$, then $\det(\mathbf{C}) = \det(\mathbf{A})\det(\mathbf{B})$. Thus if the columns (or rows) of $\mathbf{A}$ and $\mathbf{B}$ are linearly independent, then the columns (or rows) of $\mathbf{C}$ are also.

Linear Dependence & Vector Functions

- Now consider vector functions $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$, where

$$\mathbf{x}^{(k)}(t) = \begin{pmatrix} x_1^{(k)}(t) \\ x_2^{(k)}(t) \\ \vdots \\ x_m^{(k)}(t) \end{pmatrix}, \quad k = 1, 2, \ldots, n, \qquad t \in I = (\alpha, \beta)$$

- As before, $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$ is **linearly dependent** on I if there exists scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) = \mathbf{0}, \text{ for all } t \in I$$

- Otherwise $\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)$ is **linearly independent** on I. See text for more discussion on this.

Eigenvalues and Eigenvectors

- The eqn. $\mathbf{Ax} = \mathbf{y}$ can be viewed as a linear transformation that maps (or transforms) $\mathbf{x}$ into a new vector $\mathbf{y}$.
- Nonzero vectors $\mathbf{x}$ that transform into multiples of themselves are important in many applications.
- Thus we solve $\mathbf{Ax} = \lambda\mathbf{x}$ or equivalently, $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.
- This equation has a nonzero solution if we choose λ such that $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$.
- Such values of λ are called **eigenvalues** of **A**, and the nonzero solutions **x** are called **eigenvectors**.

Example 4: Eigenvalues (1 of 3)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

- Solution: Choose λ such that det(**A**-λ**I**) = 0, as follows.

$$\begin{aligned}
\det(\mathbf{A} - \lambda \mathbf{I}) &= \det\left(\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \det\begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix} \\
&= (3-\lambda)(-2-\lambda) - (-1)(4) \\
&= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \\
&\Rightarrow \ \lambda = 2, \ \lambda = -1
\end{aligned}$$

Example 4: First Eigenvector (2 of 3)

- To find the eigenvectors of the matrix **A**, we need to solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ for $\lambda = 2$ and $\lambda = -1$.
- Eigenvector for $\lambda = 2$: Solve

$$\left(\mathbf{A}-\lambda\mathbf{I}\right)\mathbf{x}=\mathbf{0} \Leftrightarrow \begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this implies that $x_1 = x_2$. So

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Example 4: Second Eigenvector (3 of 3)

- Eigenvector for $\lambda = -1$: Solve

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{pmatrix} 3+1 & -1 \\ 4 & -2+1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and this implies that $x_2 = 4x_1$ So

$$\mathbf{x}^{(2)} = \begin{pmatrix} x_1 \\ 4x_1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \; c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Normalized Eigenvectors

- From the previous example, we see that eigenvectors are determined up to a nonzero multiplicative constant.
- If this constant is specified in some particular way, then the eigenvector is said to be **normalized**.
- For example, eigenvectors are sometimes normalized by choosing the constant so that $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = 1$.

Algebraic and Geometric Multiplicity

- In finding the eigenvalues λ of an n x n matrix **A**, we solve $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$.
- Since this involves finding the determinant of an n x n matrix, the problem reduces to finding roots of an nth degree polynomial.
- Denote these roots, or eigenvalues, by $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- If an eigenvalue is repeated m times, then its **algebraic multiplicity** is m.
- Each eigenvalue has at least one eigenvector, and a eigenvalue of algebraic multiplicity m may have q linearly independent eigevectors, $1 \leq q \leq m$, and q is called the **geometric multiplicity** of the eigenvalue.

Eigenvectors and Linear Independence

- If an eigenvalue λ has algebraic multiplicity 1, then it is said to be **simple**, and the geometric multiplicity is 1 also.
- If each eigenvalue of an n x n matrix **A** is simple, then **A** has n distinct eigenvalues. It can be shown that the n eigenvectors corresponding to these eigenvalues are linearly independent.
- If an eigenvalue has one or more repeated eigenvalues, then there may be fewer than n linearly independent eigenvectors since for each repeated eigenvalue, we may have $q < m$. This may lead to complications in solving systems of differential equations.

Example 5: Eigenvalues (1 of 5)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- Solution: Choose λ such that det(**A**-λ**I**) = 0, as follows.

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} \\ &= -\lambda^3 + 3\lambda + 2 \\ &= (\lambda - 2)(\lambda + 1)^2 \\ &\Rightarrow \; \lambda_1 = 2, \; \lambda_2 = -1, \; \lambda_2 = -1 \end{aligned}$$

Example 5: First Eigenvector (2 of 5)

- Eigenvector for $\lambda = 2$: Solve $(\mathbf{A}\text{-}\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{rrrl} 1x_1 & & -1x_3 & = 0 \\ & 1x_2 & -1x_3 & = 0 \\ & & 0x_3 & = 0 \end{array}$$

$$\to \mathbf{x}^{(1)} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ c \text{ arbitrary} \to \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example 5: 2^{nd} and 3^{rd} Eigenvectors (3 of 5)

- Eigenvector for $\lambda = -1$: Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{rrrl} 1x_1 & +1x_2 & +1x_3 & =0 \\ & 0x_2 & & =0 \\ & & 0x_3 & =0 \end{array}$$

$$\to \mathbf{x}^{(2)} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{ where } x_2, x_3 \text{ arbitrary}$$

$$\to \text{choose } \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Example 5: Eigenvectors of $\mathbf{A}$ (4 of 5)

- Thus three eigenvectors of **A** are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

  where $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ correspond to the double eigenvalue $\lambda = -1$.
- It can be shown that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ are linearly independent.
- Hence **A** is a 3 x 3 **symmetric matrix** ($\mathbf{A} = \mathbf{A}^T$) with 3 real eigenvalues and 3 linearly independent eigenvectors.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Example 5: Eigenvectors of **A** (5 of 5)

- Note that we could have we had chosen

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{x}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

- Then the eigenvectors are orthogonal, since

$$\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = 0, \quad \left(\mathbf{x}^{(1)}, \mathbf{x}^{(3)}\right) = 0, \quad \left(\mathbf{x}^{(2)}, \mathbf{x}^{(3)}\right) = 0$$

- Thus **A** is a 3 x 3 symmetric matrix with 3 real eigenvalues and 3 linearly independent orthogonal eigenvectors.

Hermitian Matrices

- A **self-adjoint**, or **Hermitian** matrix, satisfies $\mathbf{A} = \mathbf{A}^*$, where we recall that $\mathbf{A}^* = \overline{\mathbf{A}}^T$.
- Thus for a Hermitian matrix, $a_{ij} = \overline{a}_{ji}$.
- Note that if **A** has real entries and is symmetric (see last example), then **A** is Hermitian.
- An n x n Hermitian matrix **A** has the following properties:
  - All eigenvalues of **A** are real.
  - There exists a full set of n linearly independent eigenvectors of **A**.
  - If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues of **A**, then $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
  - Corresponding to an eigenvalue of algebraic multiplicity m, it is possible to choose m mutually orthogonal eigenvectors, and hence A has a full set of n linearly independent orthogonal eigenvectors.