1. Discuss whether following sets are open or closed. Determine the interiors, closures and boundaries of each sets.

- a) (1,2) in \mathbb{R} .
- b) [2,3] in \mathbb{R} .
- c) $\bigcap_{n=1}^{\infty} [-1, \frac{1}{n})$ in \mathbb{R} .
- d) $(0,1) \cap \mathbb{Q}$ in \mathbb{R} .

2. Let A be a subset of a metric space (M, d). Define

 $\check{A} := \{a \in A : a \text{ is an interior point of } A\} \text{ and } int(A) := \bigcup \{O \subseteq A : O \text{ is open in } M\}.$

Show that

- a) $\check{A} = int(A)$
- b) A is open if and only if A = int(A).

3. Let A be a subset of \mathbb{R}^n . Show that

- a) $int(A) \subset A \subset \overline{A}$
- b) A is closed if and only if $A = \overline{A}$.

4.Let $A \subset \mathbb{R}$ be nonempty and bounded above. Let $x = \sup(A)$. Show that $x \in \overline{A}$ and $x \in \partial A$.

5. For a subset A of a metric space, show that $x \in \overline{A}$ if and only if

$$d(x, A) = \inf\{d(x, y) : y \in A\} = 0.$$

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6. Prove that an element $x \in (M, d)$ is an accumulation point of A if and only if there is a sequence $\{x_n\} \subset A \setminus \{x\}$ such that $\lim_{n \to \infty} x_n = x$.

7. Prove that a sequence (x_n) in a normed space is a Cauchy sequence if and only if for every neighborhood U of 0 there is an N such that m, n > N implies that $x_n - x_m \in U$.

Note that A **neighborhood** of a point in a normed space is an open set containing that point.

8. Identify \mathbb{R}^{n+m} with $\mathbb{R}^n \times \mathbb{R}^m$. Show that $A \subset \mathbb{R}^{n+m}$ is open if and only if for each $(x,y) \in A$ with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, there exists open sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ with $x \in U$, $y \in V$ such that $U \times V \subset A$. Deduce that product of open sets is open.

Hint: Consider $(x,y) \in A$ and the disk $D((x,y);\epsilon)$. Define D_x to be the projection of $D((x,y);\epsilon)$ onto its first n components and D_y be the projection onto the last m components.

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